Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 12, NUMBER 4 (2009) ISSN 1311-0454

ON THE HARDY INEQUALITY ON $L^{p(\cdot)}(R^n)$ SPACES

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Abstract

The Hardy type inequality

$$\left\| \frac{f(x)}{|x|^\alpha} \right\|_{p(\cdot)} \leq C \|(-\triangle)^{\alpha/2} f\|_{p(\cdot)}$$

is proved for the spaces $L^{p(\cdot)}(R^n)$ with variable exponent $p(\cdot)$ in the case $p(\cdot) \in VMO^{1/|\log|}(R^n), 1 < p_- \leq p_+ < n/\alpha$, and is constant outside some ball.

Mathematics Subject Classification: 47B38, 42B35, 46E35

Key Words and Phrases: Hardy inequality, Riesz potential, Lebesgue space with variable exponent

1. Introduction

Let $p(\cdot) : R^n \mapsto [1, +\infty)$ be a measurable function. Denote by $L^{p(\cdot)}(R^n)$ the space of functions f such that for some $\lambda > 0$,

$$\int_{R^n} |f(x)/\lambda|^{p(x)} dx < \infty$$

with the norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{R^n} |f(x)/\lambda|^{p(x)} dx \leq 1 \right\}.$$

The Lebesgue spaces $L^{p(\cdot)}(R^n)$ with variable exponent and the corresponding variable Sobolev spaces $W^{k,p(\cdot)}$ are of interest for their applications to modelling problems in physics, and to the study of variational integrals and partial differential equations.

The Hardy-Littlewood maximal operator M is defined on the space of locally integrable functions on R^n by

$$Mf(x) = \sup \frac{1}{|Q|} \int_Q |f(t)|\, dt,$$

where the supremum is taken over all cubes Q containing x ($|Q|$ denotes the Lebesgue measure of the set Q).

Let $\mathcal{B}(R^n)$ be the class of all exponents $p(\cdot)$, $1 < p_- \le p_+ < \infty$ ($p_- = \operatorname{essinf}_{x\in R^n} p(x)$, $p_+ = \operatorname{esssup}_{x\in R^n} p(x)$) for which the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(R^n)$.

By $\mathcal{P}(R^n)$ we denote the set of exponents $p(\cdot)$, $1 < p_- \le p_+ < \infty$ satisfying the conditions

$$|p(x) - p(y)| \le \frac{c}{-\log(|x-y|)}, \quad |x-y| \le 1/2, \tag{1}$$

$$|p(x) - p(y)| \le \frac{c}{\log(e+|x|)}, \quad |y| > |x|. \tag{2}$$

Let

$$\gamma(f,r) = \sup_{|Q|\le r} \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \tag{3}$$

where f_Q is an integral average of the function f on Q. We denote by $VMO^{1/|\log|}(R^n)$ the set of all function f for which $\gamma(f,r) = o(1/|\log(r)|)$, $r \to 0$.

By $\widetilde{\mathcal{P}}(R^n)$ we denote the set of exponents $p(\cdot) \in VMO^{1/|\log|}(R^n)$, $1 < p_- \le p_+ < \infty$ witch are constants outside some ball. Note that there exist noncontinuous exponents belonging to $\widetilde{\mathcal{P}}(R^n)$ (see [8]).

Diening [4] proved that if $p(\cdot)$ satisfies the condition (1) and if $p(\cdot)$ is a constant outside some compact set, then $p(\cdot) \in \mathcal{B}(R^n)$. Cruz-Uribe, Fiorenza and Neugebauer [2] and Nekvinda [17] proved that if $p(\cdot) \in \mathcal{P}(R^n)$ then $p(\cdot) \in \mathcal{B}(R^n)$. On the other hand, they are not necessary for $p(\cdot) \in \mathcal{B}(R^n)$. In [14], Lerner established that there exist discontinuous functions $p(\cdot) \in \mathcal{B}(R^n)$. Diening [3] showed that $p(\cdot) \in \mathcal{B}(R^n)$ if and only if the averaging operator $T_{\mathcal{Q}} : f \mapsto \sum_{Q\in\mathcal{Q}} f_Q \chi_Q$ is uniformly bounded on $L^{p(\cdot)}(R^n)$ with respect to all families $\mathcal{Q}$ of disjoint cubes. In [11], Kopaliani provides a characterization of $\mathcal{B}(R^n)$ in terms of the Muckenhoupt-type condition $A_{p(\cdot)}$ under some restriction on the behavior of $p(\cdot)$ at infinity. Then this was used in [8] in order to give a new sufficient condition for $p(\cdot) \in \mathcal{B}(R^n)$ in terms of mean oscillation of $p(\cdot)$. Indeed, if $p(\cdot) \in \widetilde{\mathcal{P}}(R^n)$, then $p(\cdot) \in \mathcal{B}(R^n)$.

By the symbol $\widetilde{\mathcal{B}}(R^n)$ we denote the family of $p(\cdot) \in \mathcal{B}(R^n)$ such that $\alpha p(\cdot) \in \mathcal{B}(R^n)$ for all $\alpha > 1/p_-$. Lerner in [13] constructed an example showing that $p(\cdot) \in \mathcal{B}(R^n)$ does not imply $p(\cdot) \in \widetilde{\mathcal{B}}(R^n)$. Obviously, $p(\cdot) \in \mathcal{P}(R^n)$ implies that $p(\cdot)$, $p'(\cdot) \in \widetilde{\mathcal{B}}(R^n)$, where $p'(\cdot)$ is the conjugate exponent of $p(\cdot)$ $(1/p(t)) + 1/p'(t) = 1$, $t \in R^n)$. Observing that

$$\begin{aligned}&\tfrac{1}{|Q|}\int\limits_Q |p'(x) - p'_Q|dx \sim \tfrac{1}{|Q|^2}\int\limits_Q\int\limits_Q |p'(x) - p'(y)|dxdy \\ &\leq \tfrac{1}{(p_- - 1)^2}\tfrac{1}{|Q|^2}\int\limits_Q\int\limits_Q |p(x) - p(y)|dxdy \sim \tfrac{1}{|Q|}\int\limits_Q |p(x) - p_Q|dx,\end{aligned}$$

we obtain $p(\cdot)$, $p'(\cdot) \in \widetilde{\mathcal{B}}(R^n)$ if $p(\cdot) \in \widetilde{\mathcal{P}}(R^n)$.

Define the Riesz potential $\mathcal{I}^\alpha f$ by

$$\mathcal{I}^\alpha f(x) = \frac{1}{\gamma_n(\alpha)}\int_{R^n} |x - y|^{-n+\alpha} f(y)dy,$$

where $\gamma_n(\alpha) = 2^\alpha \pi^{n/2}\frac{\Gamma(\alpha/2)}{\Gamma(n/2-\alpha/2)}$ and $0 < \alpha < n$.

The hypersingular integral operator of order α, $\alpha > 0$ known also as the Riesz derivative, is defined by

$$D^\alpha f(x) = \lim_{\varepsilon\to 0}\frac{1}{d_{n,l}(\alpha)}\int_{|y|>\varepsilon}\frac{(\Delta_y^l f)(x)}{|y|^{n+\alpha}}dy, \quad l > \alpha, \tag{4}$$

where $(\Delta_y^l f)(x) = \sum_{k=0}^l (-)^k C_l^k f(x - ky)$ is finite difference and $d_{n,l}(\alpha)$ is some normalizing constant (see details in [21]).

It is well known that the inverse operator $(-\triangle)^{\alpha/2}$ to the Riesz potential operator $\mathcal{I}^\alpha$ on "nice" functions has the form $(-\triangle)^{\alpha/2} f(x) = \mathcal{F}^{-1}|\xi|^\alpha \mathcal{F} f$, where $\mathcal{F}$ is the Fourier transform. In case when $p(\cdot) \in \mathcal{B}(R^n)$, $0 < \alpha < n$, $p_+ < \frac{n}{\alpha}$ the fractional power of the minus Laplace operator may be treated as the hypersingular integral (4), where the limit above is taken in the sense of convergence in the $L^{p(\cdot)}$ norm. In fact, the domain of the definition of $(-\triangle)^{\alpha/2}$ is the set $\mathcal{I}^\alpha(L^{p(\cdot)}(R^n))$ and we have $(-\triangle)^{\alpha/2}\mathcal{I}^\alpha f = D^\alpha \mathcal{I}^\alpha f = f$ (see [1]).

Let $p(\cdot) \in \mathcal{B}(R^n)$ and $0 < \alpha < n$. The Bessel potential space with variable exponent $L^{\alpha,p(\cdot)}(R^n)$ (see [1]) is defined by norm

$$\|f\|_{\alpha,p(\cdot)} = \|f\|_{p(\cdot)} + \|(-\triangle)^{\alpha/2} f\|_{p(\cdot)}.$$

We prove the following theorem.

Theorem 1.1. *Suppose $p(\cdot) \in \mathcal{B}(R^n)$ such that $p'(\cdot) \in \widetilde{\mathcal{B}}(R^n)$ and $0 < \alpha < \frac{n}{p_+}$. Then there exists a constant C such that for $f \in L^{\alpha,p(\cdot)}(R^n)$*

$$\left\|\frac{f(x)}{|x|^\alpha}\right\|_{p(\cdot)} \leq C\|(-\triangle)^{\alpha/2} f\|_{p(\cdot)}. \tag{5}$$

In the case $p(\cdot) = \text{const}$, (5) is the classical Hardy inequality. There is a rich literature concerning the Hardy inequality. We refer to books by Opic and Kufner [18], Maz'ya [15] and references therein. Hardy inequalities have been studied in the variable exponent setting in case $p(\cdot) \in \mathcal{P}(R^n)$ in [7], [10], [19], [20]. If $p(\cdot) \in \mathcal{P}(R^n)$ (or $p(\cdot) \in \widetilde{\mathcal{P}}(R^n)$), then $p(\cdot),\ p'(\cdot) \in \widetilde{\mathcal{B}}(R^n)$. Note also that $\mathcal{P}(R^n)\backslash\widetilde{\mathcal{P}}(R^n) \neq \emptyset$ and $\widetilde{\mathcal{P}}(R^n)\backslash\mathcal{P}(R^n) \neq \emptyset$ (see [8]).

2. Proof of Theorem 1.1

Below for simplicity, by C there will be denoted positive constants which depend only on the dimension and the exponent $p(\cdot)$, but whose value may change at each appearance. The proof of Theorem 1.1 follows the main steps in the proof of a similar theorem in [19] (the case of bounded domain Ω and of exponent $p(\cdot)$ with property (1)).

Note that if $f \in L^{\alpha,p(\cdot)}(R^n)$, then there exists $\varphi \in L^{p(\cdot)}(R^n)$ such that $f = \mathcal{I}^\alpha\varphi$ (see [1]).

We have

$$\frac{\mathcal{I}^\alpha\varphi(x)}{|x|^\alpha} = \int\limits_{R^n} \frac{\varphi(y)}{|x-y|^{n-\alpha}|x|^\alpha}dy$$

$$= \int\limits_{|x-y|\le 4|x|} \frac{\varphi(y)}{|x-y|^{n-\alpha}|x|^\alpha}dy + \int\limits_{|x-y|>4|x|} \frac{\varphi(y)}{|x-y|^{n-\alpha}|x|^\alpha}dy$$

$$= A\varphi(x) + B\varphi(x).$$

The operator $A\varphi(x)$ admits the pointwise estimate

$$|A\varphi(x)| \le CM\varphi(x).$$

This estimate is contained in [19]. We briefly outline the proof in different way:

$$|A\varphi(x)| \le \sum_{j\le 1} \int\limits_{2^j|x|<|x-y|\le 2^{j+1}|x|} \frac{|\varphi(y)|}{|x-y|^n|x|^\alpha}dy$$

$$\le \sum_{j\le 1} \int\limits_{2^j|x|<|x-y|\le 2^{j+1}|x|} \frac{|\varphi(y)|}{2^{j(n-\alpha)}|x|^{n-\alpha}|x|^\alpha}dy$$

$$= \sum_{j\le 1} \int\limits_{2^j|x|<|x-y|\le 2^{j+1}|x|} \frac{|\varphi(y)|}{2^{jn}|x|^n}2^{\alpha j}dy$$

$$\le C\sum_{j\le 1} \frac{2^{\alpha j}}{2^{jn}|x|^n} \int\limits_{|x-y|\le 2^{j+1}|x|} |\varphi(y)| \le C\sum_{j\le 1} 2^{j\alpha}M\varphi(x) \le C\,M\varphi(x).$$

Since $p(\cdot) \in \mathcal{B}(R^n)$ we have
$$\|A\varphi\|_{p(\cdot)} \leq C\|M\varphi\|_{p(\cdot)} \leq C\|\varphi\|_{p(\cdot)}. \tag{6}$$
It remains to prove the boundedness of the operator B. Obviously, $|x-y| > 4|x|$ implies that $|y| \geq |x-y| - |x| \geq 4|x| - |x| = 3|x|$ and we have
$$|x-y| \geq |y| - |x| \geq |y| - \frac{|y|}{3} = \frac{2|y|}{3}.$$
The operator $B\varphi(x)$ admits the pointwise estimate
$$|B\varphi(x)| \leq \int\limits_{|x-y|>4|x|} \frac{|\varphi(y)|}{|x-y|^{n-\alpha}|x|^{\alpha}} dy \leq C \int\limits_{|y|\geq|3x|} \frac{|\varphi(y)|}{|y|^{n-\alpha}|x|^{\alpha}} dy = C \cdot B_1\varphi(x).$$
For any $g \in L^{p'(\cdot)}(R^n)$ we have
$$< B_1\varphi,\, g >= \int\limits_{R^n} \int\limits_{|y|\geq 3|x|} \frac{|\varphi(y)|}{|y|^{n-\alpha}|x|^{\alpha}} dy \cdot g(x) dx$$
$$= \int\limits_{R^n} \frac{1}{|y|^{n-\alpha}} \int\limits_{|x|\leq\frac{|y|}{3}} \frac{g(x)}{|x|^{\alpha}} dx \cdot |\varphi(y)| dy =< Tg(y),\, |\varphi(y)| >,$$
where the operator Tg has the form
$$Tg(y) = \frac{1}{|y|^{n-\alpha}} \int\limits_{|x|\leq\frac{|y|}{3}} \frac{g(x)}{|x|^{\alpha}} dx.$$
Fix l such that $p_+ < l$ and $\alpha l < n$. Using Hölder inequality we obtain
$$|Tg(y)| \leq \frac{1}{|y|^{n-\alpha}} \left(\int\limits_{|x|\leq\frac{|y|}{3}} |g(y)|^{l'} dx \right)^{1/l'} \left(\int\limits_{|x|\leq\frac{|y|}{3}} \frac{1}{|x|^{\alpha l}} dx \right)^{1/l}$$
$$\leq C \frac{1}{|y|^{n-\alpha}} \left(\int\limits_{|x|\leq\frac{|y|}{3}} |g(y)|^{l'} dx \right)^{1/l'} |y|^{(n-\alpha l)/l}$$
$$\leq C \frac{1}{|y|^{n/l'}} \left(\int\limits_{|x|\leq\frac{|y|}{3}} |g(y)|^{l'} dx \right)^{1/l'} \leq C(M(|g|^{l'})^{1/l'}(x)).$$
Note that $p'(t) \geq p'_- > l'$. On the other hand $p'(\cdot) \in \widetilde{\mathcal{B}}(R^n)$ and we have
$$\|Tg\|_{p'(\cdot)} \leq C\|(M(|g|^{l'}))^{1/l'}\|_{p'(\cdot)} = C\|(M(|g|^{l'}))\|^{1/l'}_{p'(\cdot)/l'}$$
$$\leq C\||g|^{l'}\|^{1/l'}_{p'(\cdot)/l'} = C\||g|\|_{p'(\cdot)}.$$
Therefore,
$$\|B_1\varphi\|_{p(\cdot)} = \sup_{\|g\|_{p'(\cdot)}\leq 1} |< B_1\varphi,\, g >| = \sup_{\|g\|_{p'(\cdot)}\leq 1} |< Tg(y),\, |\varphi(y)| >|$$
$$\leq 2\|Tg\|_{p'(\cdot)}\|\varphi\|_{p(\cdot)} \leq C\|g\|_{p'(\cdot)}\|\varphi\|_{p(\cdot)} \leq C\|\varphi\|_{p(\cdot)}.$$
Using last estimate and (6) we obtain desired result. ∎

3. Some further results

Let Ω is an open set in R^n. Given $0 < \alpha < n$, define the fractional integral operator $\mathcal{I}_\alpha^\Omega$ by

$$\mathcal{I}_\alpha^\Omega f(x) = \int_\Omega \frac{f(x)}{|x-y|^{n-\alpha}} dy.$$

For a bounded domain Ω we define $VMO^{1/|\log|}(\Omega)$ as in case R^n taking $f \in L^1(\Omega)$ and replacing in (3) the cube Q by the intersection $\Omega \cap Q$. By $\widetilde{\mathcal{P}}(\Omega)$ we denote the set of exponents $p(\cdot) \in VMO^{1/|\log|}(\Omega)$ such that $1 < p_- \le p_+ < \infty$ ($p_- = \operatorname{essinf}_{x\in\Omega} p(x)$, $p_+ = \operatorname{esssup}_{x\in\Omega} p(x)$).

THEOREM 3.1. *Let Ω be an open bounded set with Lipschitz boundary and $p(\cdot) \in \widetilde{\mathcal{P}}(\Omega)$ and $p_+ < n/\alpha$, $0 < \alpha < n$. Then there exists a constant C such that for $f \in L^{p(\cdot)}(\Omega)$:*

$$\left\| \frac{\mathcal{I}_\alpha^\Omega f}{|x|^\alpha} \right\|_{p(\cdot)} \le C \|f\|_{p(\cdot)}. \tag{7}$$

P r o o f. Note that if $p(\cdot) \in VMO^{1/|\log|}(\Omega)$, $1 < p_- \le p_+ < \infty$, then there exists $\widetilde{p}(\cdot) \in VMO^{1/|\log|}(R^n)$ with property $1 < \widetilde{p}_- \le \widetilde{p}_+ < \infty$, $p(x) = \widetilde{p}(x)$ for $x \in \Omega$, and $\widetilde{p}(\cdot)$ is constant outside some ball (see [8]). Let $\widetilde{f}$ be the zero extension of a function f defined on Ω. By Theorem 1.1 we have

$$\left\| \frac{\mathcal{I}_\alpha^\Omega f}{|x|^\alpha} \right\|_{p(\cdot)} = \left\| \frac{\mathcal{I}_\alpha \widetilde{f}}{|x|^\alpha} \chi_\Omega \right\|_{\widetilde{p}(\cdot)} \le C \|\widetilde{f}\|_{\widetilde{p}(\cdot)} = \|f\|_{p(\cdot)}.$$

■

Note that (7) type inequality has been studied by Samko in [19] under condition (1). Recently Rafeiro and Samko proved that when $p(\cdot)$ satisfies condition (1) and Ω is a bounded domain with property $R^n \backslash \overline{\Omega}$ has the cone property, then the validity of the Hardy type (7) inequality is equivalent to certain property of the domain Ω expressed in terms of α and χ_Ω (see [23]).

Let f be a function on some open set Ω such that its distributional derivatives of order m are locally summable. Under $\nabla^m f$ we mean the gradient of f of order m, i.e.

$$\nabla^m f = \{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_1},\ \alpha_1 + \cdots + \alpha_n = m\},$$

where ∂_{x_j} is partial derivative. The Euclidian length of $\nabla^m f$ will be denoted by $|\nabla^m f|$.

The multidimensional Hardy inequality of the form

$$\int_\Omega |f(x)|^p d(x)^{-p+a} \le C \int_\Omega |\nabla f(x)|^p d(x)^a dx \tag{8}$$

where $d(x) = \operatorname{dis}(x, \Omega)$ and $f \in C_0^\infty(\Omega)$ appeared in [16] for bounded domain $\Omega \subset R^n$ with Lipschitz boundary and $1 < p < \infty$ and $a < p - 1$.

Lewis [12] and Wannebo [24] gave independent proofs that the Hardy inequality (8) holds in a proper open subset Ω of R^n provided its complement is fat enough with respect to certain capacity. Hajlasz [6] and Kinnunen and Martio [9] obtained the pointwise inequality

$$|f(x)| \le d(x)\mathcal{M}|\nabla f|(x), \tag{9}$$

where $\mathcal{M}$ is kind of maximal function depending on the distance of x to the boundary. This pointwise inequality combined with the boundedness of Hardy-Littlewood maximal operator implies Hardy's inequality.

Harjulehto, Hästö and Koskenoja in [7] obtained the estimate

$$\left\| \frac{f(x)}{d(x)^{1-a}} \right\|_{p(\cdot)} \le C \| |\nabla f(x)| d(x)^a \|_{p(\cdot)}, \ f \in C_0^\infty(\Omega),$$

making use of the approach of [6], under the assumption that a is sufficiently small $0 \le a < a_0$ and Ω is an open bounded subset of R^n such that for some constant $b > 0$

$$|B(z, r) \cap \Omega^c| \ge b|B(z, r)|$$

for every $z \in \partial\Omega$ and $r > 0$.

Edmunds and Rákosník in [5] extended (9) for gradients of higher order. Indeed, if Ω is an open bounded set that $R^n \backslash \Omega$ is uniformly p-fat for some p, $1 < p < \infty$, then there exists a constant $c > 0$ such that every function $f \in C_0^\infty(\Omega)$ satisfies the inequality

$$|f(x)| \le c d(x)^k [M(|\nabla^k f|^p(x))]^{1/p}, \ \ x \in \Omega. \tag{10}$$

(In the context of maximal operators we extend the function on which the maximal operator is acting by 0 outside its original domain of definition.)

The requirement on Ω is not very restrictive, since all Lipschitz domains or domains satisfying the exterior cone condition are uniformly p-fat for every p, $1 < p < \infty$.

Theorem 3.2. *Let the exponent $p(\cdot) \in \widetilde{P}(\Omega)$, where Ω be a bounded domain with Lipschitz boundary. Then there exists a constant C such that for $f \in C_0^\infty(\Omega)$*

$$\left\| \frac{f(x)}{d(x)^k} \right\|_{p(\cdot)} \le C \| |\nabla^k f| \|_{p(\cdot)}. \tag{11}$$

P r o o f. Fix p_0 with property $1 < p_0 < p_-$. Using (10), we obtain

$$\left\| \frac{f(x)}{d(x)^k} \right\|_{p(\cdot)} \le C \| [M(|\nabla^k f|^{p_0})]^{1/p_0} \|_{p(\cdot)}$$

$$= C \| M(|\nabla^k f|^{p_0}) \|_{p(\cdot)/p_0}^{1/p_0} \le C \| |\nabla^k f|^{p_0} \|_{p(\cdot)/p_0}^{1/p_0} = \| |\nabla^k f| \|_{p(\cdot)}.$$

■

COROLLARY 3.1. *Let the exponent $p(\cdot) \in \widetilde{P}(\Omega)$, where Ω be a bounded domain with Lipschitz boundary. Then there exists a constant C and a_0 such that the inequality*

$$\left\| \frac{f(x)}{d(x)^{k-a}} \right\|_{p(\cdot)} \le C \| |\nabla^k f| d(x)^a \|_{p(\cdot)} \tag{12}$$

holds for all $f \in C_0^\infty(\Omega)$ and all $0 \le a < a_0$.

P r o o f. Assume that $0 < a < 1$ and $k = 1$. We use the standard bootstrapping procedure (see [5]) to deal with this case. We set $g(x) = |f(x)| d(x)^a$. Since the Lipschits constant of $d(x)$ equals 1, we obtain

$$|\nabla g(x)| \le |\nabla f(x)| d(x)^a + a |f(x)| d(x)^{a-1} \text{ for a.e. } x \in \Omega.$$

Applying inequality (11) to g we obtain

$$\left\| \frac{f(x)}{d(x)^{1-a}} \right\|_{p(\cdot)} \le C (\| |\nabla f| d(x)^a \|_{p(\cdot)} + a \| f(x) d(x)^{a-1} \|_{p(\cdot)}).$$

Whenever $Ca < 1$ this yields (12) for $k = 1$.

Let $k > 1$ and suppose that the inequality (12) holds for $j = 1, 2, ..., k-1$ and $0 \le a < a_0$. Let ϱ be the regularized distance equivalent to d and satisfying the estimate

$$|\nabla^j \varrho(x)| \le c_j d(x)^{1-j}, \ \ x \in \Omega, \ j = 1, 2, ...,$$

(see [22]). Set $g(x) = f(x) \varrho(x)^a$. Then

$$|\nabla^k g(x)| \le |\nabla^k f(x)| \varrho(x)^a + a \sum_{j=1}^{k} Q_j(a) |\nabla^{k-j} f(x)| \varrho(x)^{a-j},$$

where Q_j are polynomials of degree j. Thus, we have

$$\left\| \frac{f(x)}{d(x)^{k-a}} \right\|_{p(\cdot)} \le C \left\| \frac{g(x)}{\varrho(x)^k} \right\|_{p(\cdot)}$$

$$\le C \| |\nabla^k f(x)| \varrho(x)^a \|_{p(\cdot)} + aC \sum_{j=1}^{k} |Q_j(a)| \cdot \| |\nabla^{k-j} f(x)| \varrho(x)^{a-j} \|_{p(\cdot)}$$

$$\le aC\left\|\frac{f(x)}{\varrho(x)^{k-a}}\right\|_{p(\cdot)}+C\||\nabla^k f(x)|\varrho(x)^a\|_{p(\cdot)}+aC\sum_{j=1}^{k-1}\cdot\||\nabla^{k-j}f(x)|\varrho(x)^{a-j}\|_{p(\cdot)}$$

$$\le aC\left\|\frac{f(x)}{d(x)^{k-a}}\right\|_{p(\cdot)}+c\||\nabla^k f(x)|\varrho(x)^a\|_{p(\cdot)},$$

and the inequality (12) holds. ■

Acknowledgements. The author expresses his thanks to the referees for carefully checking the manuscript, and for their valuable comments and suggestions.

This paper was supported by Grant GNSF/ST 08/3-385.

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Received: July 8, 2009